ON PSEUDO B-WEYL OPERATORS AND GENERALIZED DRAZIN INVERTIBILITY FOR OPERATOR MATRICES

H. ZARIOUH AND H. ZGUITTI

Abstract. We introduce a new class which generalizes the class of B-Weyl operators. We say that $T \in L(X)$ is pseudo B-Weyl if $T = T_1 \oplus T_2$ where T_1 is a Weyl operator and T_2 is a quasi-nilpotent operator. We show that the corresponding pseudo B-Weyl spectrum $\sigma_{pBW}(T)$ satisfies the equality $\sigma_{pBW}(T) \cup [\mathcal{S}(T) \cap \mathcal{S}(T^*)] = \sigma_{gD}(T)$; where $\sigma_{gD}(T)$ is the generalized Drazin spectrum of $T \in L(X)$ and $\mathcal{S}(T)$ (resp., $\mathcal{S}(T^*)$) is the set where T (resp., T^*) fails to have SVEP. We also investigate the generalized Drazin invertibility of upper triangular operator matrices by giving sufficient conditions which assure that the generalized Drazin spectrum or the pseudo B-Weyl spectrum of an upper triangular operator matrices is the union of its diagonal entries spectra.

1. Introduction and Preliminaries

Let X and Y be Banach spaces and let $L(X, Y)$ denote the algebra of all bounded linear operators from X to Y. We shall write $L(X)$ for the algebra $L(X, X)$. For $T \in L(X)$, by T^*, $\mathcal{N}(T)$, $\mathcal{R}(T)$, $\sigma(T)$, $\sigma_l(T)$, $\sigma_r(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$ and $\sigma_s(T)$, we denote respectively, the adjoint of T, the null space, the range, the spectrum of T, the left spectrum of T, the right spectrum of T, the point spectrum of T, the approximate point spectrum of T and the surjective spectrum of T.

A bounded linear operator $T \in L(X)$ is said to have the *single-valued extension property* (SVEP for short) at $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the constant function $f \equiv 0$ is the only analytic solution of the equation $(T - \mu I)f(\mu) = 0 \quad \forall \mu \in U_\lambda$. We denote by $\mathcal{S}(T)$ the open set of $\lambda \in \mathbb{C}$ where T fails to have SVEP at λ, and we say that T has SVEP if $\mathcal{S}(T) = \emptyset$. It is easy to see that $\mathcal{S}(T) \subset \sigma_p(T)$ (See [21] for more details about this spectral property). According to [22, Lemma3] we have

$$\sigma(T) = \mathcal{S}(T) \cup \sigma_s(T)$$

and in particular $\sigma_s(T)$ contains the topological boundary of $\mathcal{S}(T)$. Moreover, it is obvious that T has SVEP at every point $\lambda \in \mathrm{iso}\sigma(T)$. Henceforth, the symbol $\mathrm{iso}\Lambda$ means isolated points of a given subset Λ of $\mathbb{C}$ and $\mathrm{acc}\Lambda$ denotes the set of all points of accumulation of Λ.

$T \in L(X)$ is called an *upper semi-Fredholm* (resp., *lower semi-Fredholm*) if $\mathcal{R}(T)$ is closed and $n(T) := \dim \mathcal{N}(T) < +\infty$ (resp., $d(T) := \mathrm{codim}\mathcal{R}(T) < +\infty$). If T is either upper or lower semi-Fredholm then T is called a *semi-Fredholm* operator. The *index* of a semi-Fredholm operator T is defined by $\mathrm{ind}(T) = n(T) - d(T)$. $T \in L(X)$ is called a *Fredholm* operator if both $n(T)$ and $d(T)$ are finite, and is

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called a *Weyl* operator if it is a Fredholm of index zero. The *essential spectrum* of T is defined by $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not a Fredholm operator$\}$, and the *Weyl spectrum* of T is defined by $\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not a Weyl operator$\}$. Let $\mathcal{F}(X)$ denote the ideal of finite rank operators in $L(X)$. Then it is well known that

$$\sigma_W(T) = \bigcap_{F \in \mathcal{F}(X)} \sigma(T + F).$$

Recall that an operator $T \in L(X)$ is said to be *semi-regular*, if $\mathcal{R}(T)$ is closed and $\mathcal{N}(T^n) \subseteq \mathcal{R}(T)$, for all $n \in \mathbb{N}$, see for example [28]. In addition, it was proved in [18] that given a semi-Fredholm operator $T \in L(X)$, there exist two closed T-invariant subspaces X_1, X_2 such that $X = X_1 \oplus X_2$, $T|X_1$ is nilpotent and $T|X_2$ is semi-regular. This decomposition is known as the *Kato decomposition*, and the operators satisfying these conditions, which were characterized in [24], are said to be the *quasi-Fredholm operators*.

Berkani gave a generalization of Fredholm operators as follows: for each nonnegative integer n define $T_{[n]}$ to be the restriction of T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_{[0]} = T$). If for some n, $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is a Fredholm operator then T is called a *B-Fredholm* operator. T is said to be a *B-Weyl* operator if $T_{[n]}$ is a Fredholm operator of index zero (see [6]). The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}.$$

On the other hand, and according to [4, Proposition 2.6], a B-Fredholm operator is quasi-Fredholm; what is more, according to [4, Theorem 2.7], if $T \in L(X)$ is B-Fredholm, then there exist two closed T-invariant subspaces X_1, X_2 such that $X = X_1 \oplus X_2$, $T|X_1$ is Fredholm and $T|X_2$ is nilpotent (see also [29, Theorem 7]).

An operator $T \in L(X)$ is said to be a *Drazin invertible* if there exists a positive integer k and an operator $S \in L(X)$ such that

$$ST = TS, \quad T^{k+1}S = T^k \text{ and } S^2T = S.$$

It is well known that T is Drazin invertible if and only if $T = U \oplus V$; where U is an invertible operator and V is a nilpotent one (see [23, Corollary 2.2]). The *Drazin spectrum* of $T \in L(X)$ is defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}.$$

In [6] it is shown that

$$\sigma_{BW}(T) = \bigcap_{F \in \mathcal{F}(X)} \sigma_D(T + F).$$

From [6, Lemma 4.1], T is a B-Weyl operator if and only if $T = F \oplus N$, where F is a Weyl operator and N is a nilpotent operator. Hence $\sigma_{BW}(T) \subset \sigma_D(T)$. The defect set $\sigma_D(T) \setminus \sigma_{BW}(T)$ has been characterized in [1, 33] as follows:

$$\sigma_{BW}(T) \cup [\mathcal{S}(T) \cap \mathcal{S}(T^*)] = \sigma_D(T). \tag{1.1}$$

Quasi-Fredholm operators were generalized to pseudo Fredholm operators. In fact, $T \in L(X)$ is said to be a *pseudo Fredholm* operator if there exist two closed T-invariant subspaces X_1, X_2 such that $X = X_1 \oplus X_2$, $T|X_1$ is quasi-nilpotent and $T|X_2$ is semi-regular. This decomposition is called the *generalized Kato decomposition*, see [26, 27].

Following Koliha [19], an operator $T \in L(X)$ is *generalized Drazin invertible* if and only if $0 \notin \text{acc}\sigma(T)$, which is also equivalent to the fact that $T = T_1 \oplus T_2$; where T_1 is invertible and T_2 is quasi-nilpotent. The *generalized Drazin spectrum* of $T \in L(X)$ is defined by

$$\sigma_{gD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not generalized Drazin invertible}\}.$$

For more details about generalized Drazin invertibility, we refer the reader to [10, 19, 20]. It is not difficult to see that $\sigma_D(T) = \sigma_{gD}(T) \cup \text{iso}\sigma_D(T)$. The inclusion $\sigma_{gD}(T) \subset \sigma_D(T)$ may be strict. Indeed, let T defined on $l^2(\mathbb{N})$ by

$$T(x_1, x_2, x_3, \ldots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \ldots),$$

then it is clear that T is quasi-nilpotent with infinite ascent. Hence $\sigma_{gD}(T) = \emptyset$ and $\sigma_D(T) = \{0\}$.

More recently, B-Fredholm operators were generalized to pseudo B-Fredholm operators. Precisely, $T \in L(X)$ is said to be a *pseudo B-Fredholm* operator if there exist two closed T-invariant subspaces X_1, X_2 such that $X = X_1 \oplus X_2$, $T|X_1$ is quasi-nilpotent and $T|X_2$ is Fredholm, see [7].

As a continuation in this direction, in the second section of the present work, we generalize the B-Weyl operators and then the Weyl operators to pseudo B-Weyl operator. $T \in L(X)$ will be said to be pseudo B-Weyl operator if T can be written as $T = T_1 \oplus T_2$; where T_1 is Weyl operator and T_2 is quasi-nilpotent. The corresponding spectrum will be denoted by $\sigma_{pBW}(T)$. Among other things, we prove that

$$\sigma_{pBW}(T) \cup [\mathcal{S}(T) \cap \mathcal{S}(T^*)] = \sigma_{gD}(T).$$

We prove also that

$$\bigcap_{F \in \mathcal{F}(X)} \sigma_{gD}(T + F) \subset \sigma_{pBW}(T).$$

In the third section, we investigate the generalized Drazin spectrum of upper triangular operator matrices $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$, where $A \in L(X)$, $B \in L(Y)$ and $C \in L(Y, X)$. After remarking that the inclusion $\sigma_{gD}(M_C) \subset \sigma_{gD}(A) \cup \sigma_{gD}(B)$ is proper, we investigate the defect set $[\sigma_{gD}(A) \cup \sigma_{gD}(B)] \setminus \sigma_{gD}(M_C)$ in connection with local spectral theory. Precisely, we prove that $\sigma_{gD}(M_C) \cup [\mathcal{S}(A^*) \cap \mathcal{S}(B)] = \sigma_{gD}(A) \cup \sigma_{gD}(B)$ for all $C \in L(Y, X)$, and we give sufficient conditions on A and B which ensure the equality $\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$. We also investigate the largest set of operators $C \in L(Y, X)$ for which the last equality holds for all $A \in L(X)$ and $B \in L(Y)$.

2. On pseudo B-Weyl operators

Definition 2.1. Let $T \in L(X)$. We say that T is *pseudo B-Weyl* if there exist two closed T-invariant subspaces X_1, X_2 such that $X = X_1 \oplus X_2$, $T|X_1$ is a Weyl operator and $T|X_2$ is a quasi-nilpotent operator. The pseudo B-Weyl spectrum $\sigma_{pBW}(T)$ of T is defined by $\sigma_{pBW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not pseudo B-Weyl}\}$.

It is easy to see that T is pseudo B-Weyl if and only if T^* is pseudo B-Weyl. Then $\sigma_{pBW}(T) = \sigma_{pBW}(T^*)$. Let $pBW(X)$ denote the class of all pseudo B-Weyl operators. From the definition of pseudo B-Weyl operators, it is easily seen that all B-Weyl operators, all quasi-nilpotent operators and all generalized Drazin operators

are pseudo B-Weyl operators. So the class $pBW(X)$ contains $BW(X)$ the class of B-Weyl operators as a proper subclass.

Theorem 2.2. *Assume that $\mathcal{H}$ is a separable, infinite dimensional, complex Hilbert space. Then for every $T \in L(\mathcal{H})$ the following assertions are equivalent:*
i) T is in the norm closure of $pBW(\mathcal{H})$;
ii) T is in the norm closure of $BW(\mathcal{H})$.

Proof. (i) $\Longrightarrow$ (ii) because $BW(\mathcal{H}) \subset pBW(\mathcal{H})$.
(ii) $\Longrightarrow$ (i) Let $T \in pBW(\mathcal{H})$. Then $T = T_1 \oplus T_2$ where T_1 is Weyl operator and T_2 is quasi-nilpotent. Then it follows from [2] that there exists a sequence of nilpotent operators $T_{2,n}$ which converges in norm to T_2. Hence $T_1 \oplus T_{2,n}$ is a sequence of B-Weyl operators which converges in norm to T. Thus T is in the norm closure of $BW(\mathcal{H})$. □

Corollary 2.3. *Assume that $\mathcal{H}$ is a separable, infinite dimensional, complex Hilbert space. Then*

$$\overline{gBW(\mathcal{H})}^{\|\,\|} = \overline{BW(\mathcal{H})}^{\|\,\|}.$$

Recall that $T \in L(X)$ is of *finite descent* if there exists a nonnegative integer p such that $\mathcal{R}(T^p) = \mathcal{R}(T^{p+1})$.

Proposition 2.4. *Let $T \in L(X)$ with finite descent. Then T is pseudo B-Weyl if and only if T is B-Weyl.*

Proof. If T is pseudo B-Weyl , then $X = X_1 \oplus X_2$, where X_1, X_2 are closed subspaces of X, $T|X_1$ is Weyl operator and $T|X_2$ is quasi-nilpotent operator. Since T is of finite descent, then $T|X_1$ and $T|X_2$ both are of finite descent. Now $T|X_2$ is quasi-nilpotent with finite descent, then it follows from [31, Corollary 10.6] that $T|X_2$ is nilpotent operator. Thus T is B-Weyl operator by [6, Lemma 4.1]. The opposite sense is always true. □

Remark 2.5. Let T be a bilateral shift on $l^2(\mathbb{Z})$. Then T is pseudo B-Weyl if and only if T is Weyl operator or T is quasi-nilpotent operator. Indeed, if T is pseudo B-Weyl , then there exist two closed T-invariant subspaces X_1 and X_2 such that $l^2(\mathbb{Z}) = X_1 \oplus X_2$, $T|X_1$ is Weyl operator and $T|X_2$ is quasi-nilpotent operator. Let P be the projection on X_1 with $\mathcal{R}(P) = X_1$ and $\mathcal{N}(P) = X_2$. Since P commutes with T then by [30, Theorem 3] there exists some $\phi \in L^\infty(\beta)$ such that M_ϕ is similar to P. Since $P^2 = P$ then $\phi^2 = \phi$. Hence $\phi = 1$ or $\phi = 0$. Thus $P = I$ or $P = 0$. Then $X_1 = l^2(\mathbb{Z})$ or $X_1 = \{0\}$. It follows that T is Weyl or quasi-nilpotent. The converse is trivial.

It is easily seen that $\sigma_{pBW}(T) \subset \sigma_{gD}(T)$. But, in general, this inclusion is proper as we can see in the following example.

Example 2.6. Here and elsewhere S denotes the unilateral unweighted shift operator on $l^2(\mathbb{N})$ defined by

$$S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots).$$

Let $T = S \oplus S^*$. Then $\sigma_{gD}(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. As $n(T) = d(T) = 1$ then T is pseudo B-Weyl. So $0 \notin \sigma_{pBW}(T)$. This shows that the inclusion $\sigma_{pBW}(T) \subset \sigma_{gD}(T)$ is proper.

Then it is naturel to ask about the defect set $\sigma_{gD}(T) \setminus \sigma_{pBW}(T)$. Thanks to the SVEP we give a characterization of this defect set.

Theorem 2.7. *Let $T \in L(X)$. Then*

$$\sigma_{pBW}(T) \cup [\mathcal{S}(T) \cap \mathcal{S}(T^*)] = \sigma_{gD}(T).$$

Proof. Since $\sigma_{pBW}(T) \cup [\mathcal{S}(T) \cap \mathcal{S}(T^*)] \subset \sigma_{gD}(T)$ always holds, let $\lambda \in \sigma_{gD}(T) \setminus \sigma_{pBW}(T)$. Without loss of generality we can assume that $\lambda = 0$. Then $T = T_1 \oplus T_2$ on $X = X_1 \oplus X_2$ such that T_1 is Weyl operator and T_2 is quasi-nilpotent operator. Assume that $0 \notin \mathcal{S}(T) \cap \mathcal{S}(T^*)$.

Case 1. $0 \notin \mathcal{S}(T)$: since T has SVEP at 0, then T_1 also has SVEP at 0. As T_1 is Weyl operator and then is B-Weyl, it follows from [1, Theorem 2.3] that T_1 is Drazin invertible. Moreover, $0 \in \sigma(T_1)$, because in the otherwise, T_1 will be invertible and therefore T is generalized Drazin invertible, a contradiction. Hence $0 \notin \text{acc}\sigma(T_1)$. Since T_2 is quasi-nilpotent then $0 \notin \text{acc}\sigma(T)$. Thus T is generalized Drazin invertible. Which leads a contradiction.

Case 2. $0 \notin \mathcal{S}(T^*)$: the proof follows similarly. □

From Theorem 2.7, in the following corollary, we explore sufficient conditions which ensures the equalities $\sigma_{pBW}(T) = \sigma_{gD}(T)$. We point out that for the operator T defined in Example 2.6 we have $\mathcal{S}(T) = \mathcal{S}(T^*) = \mathcal{S}(S^*) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| < 1\}$ (see for instance [17, 21]). Hence $\mathcal{S}(T) \cap \mathcal{S}(T^*) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| < 1\}$.

Corollary 2.8. *Let $T \in L(X)$. If $\mathcal{S}(T) \cap \mathcal{S}(T^*) = \emptyset$, then*

$$\sigma_{pBW}(T) = \sigma_{gD}(T).$$

In particular, the equality holds if T or T^* has SVEP.

In the next proposition, we show that generalized Drazin spectrum is stable under quasi-nilpotent and finite rank commuting perturbations.

Proposition 2.9. *Let $T \in L(X)$. The the following statements hold.*
i) If $F \in \mathcal{F}(X)$ and commutes with T, then $\sigma_{gD}(T + F) = \sigma_{gD}(T)$.
ii) If $Q \in L(X)$ is a quasi-nilpotent and commutes with T, then $\sigma_{gD}(T + Q) = \sigma_{gD}(T)$.

Proof. (i) From [25, Lemma 2.1] we know that $\text{acc}\sigma(T + F) = \text{acc}\sigma(T)$. Then $\lambda \notin \text{acc}\sigma(T + F) \iff \lambda \notin \text{acc}\sigma(T)$. Hence $T + F - \lambda I$ is generalized Drazin invertible if and only if $T - \lambda I$ is generalized Drazin invertible, as desired.
(ii) Since $\sigma(T + Q) = \sigma(T)$ then $\text{acc}\,\sigma(T + Q) = \text{acc}\sigma(T)$. Thus $T + Q - \lambda I$ is generalized Drazin invertible $\iff T - \lambda I$ is. So $\sigma_{gD}(T + Q) = \sigma_{gD}(T)$. □

Theorem 2.10. *Let $R, T, U \in L(X)$ be such that $TRT = TUT$. Then*

$$\sigma_{gD}(TR) = \sigma_{gD}(UT).$$

Proof. Since $\sigma(TR)\setminus\{0\} = \sigma(UT)\setminus\{0\}$, by [9, Theorem 1], then it is enough to show that TR is generalized Drazin invertible $\iff UT$ is. Assume that $0 \notin \sigma_{gD}(TR)$, then $0 \notin \text{acc}\sigma(TR)$. Therefore $TR - \mu I$ is invertible for all small $\mu \neq 0$. Hence $UT - \mu I$ is invertible for all small $\mu \neq 0$. So $0 \notin \text{acc}\sigma(UT)$. Hence UT is generalized Drazin invertible $\iff TR$ is. □

In particular if $R = U$ we get

Corollary 2.11. *Let $R, T \in L(X)$ then*

$$\sigma_{gD}(TR) = \sigma_{gD}(RT).$$

Since the equality $\mathcal{S}(TR) = \mathcal{S}(UT)$ always holds (see [9, Theorem 9]), then it follows from Theorem 2.7 that $\sigma_{pBW}(TR) \cup [\mathcal{S}(TR) \cap \mathcal{S}(R^*T^*)] = \sigma_{pBW}(UT) \cup [\mathcal{S}(TR) \cap \mathcal{S}(R^*T^*)]$. In particular we get from last theorem that for R and $T \in L(X)$, $\sigma_{gD}(TR) = \sigma_{gD}(RT)$ and $\sigma_{pBW}(TR) \cup (\mathcal{S}(TR) \cap \mathcal{S}(R^*T^*)) = \sigma_{pBW}(RT) \cup [\mathcal{S}(TR) \cap \mathcal{S}(R^*T^*)]$.

Theorem 2.12. *Let $T \in L(X)$. Then*

$$\bigcap_{F \in \mathcal{F}(X)} \sigma_{gD}(T+F) \subset \sigma_{pBW}(T).$$

Proof. Let $\lambda \notin \sigma_{pBW}(T)$ arbitrary, then $T - \lambda I$ is pseudo B-Weyl operator. Therefore $X = X_1 \oplus X_2$ and $T - \lambda I = T_1 \oplus T_2$ relatively to this decomposition, with T_1 is Weyl operator and T_2 is quasi-nilpotent operator. By [15, Theorem 6.5.2] there exists a finite rank operator F_1 such that $T_1 + F_1$ is invertible. Let $F = F_1 \oplus 0$. Then F is a finite rank operator, $(T - \lambda I) + F = (T_1 + F_1) \oplus T_2$ is generalized Drazin invertible and $\lambda \notin \bigcap_{F \in \mathcal{F}(X)} \sigma_{gD}(T+F)$. □

We would like to finish this section with the following

Question: *Is it true that $\sigma_{pBW}(T) = \bigcap_{F \in \mathcal{F}(X)} \sigma_{gD}(T+F)$?*

3. Generalized Drazin invertibility for operator matrices

For bounded linear operators $A \in L(X), B \in L(Y)$ and $C \in L(Y, X)$, by M_C we denote the operator matrices $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ defined on $X \oplus Y$.

It is well known that, in the case of infinite dimensional, the inclusion $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$ may be strict . Hence several authors have been interested by the defect set $[\sigma_*(A) \cup \sigma_*(B)] \setminus \sigma_*(M_C)$ where σ_* runs different type spectra, see for instance [3, 8, 11, 12, 13, 14, 16, 32, 33, 34, 35, 36] and the references therein.

We begin this section by proving that the generalized Drazin spectrum of a direct sum is the union of generalized Drazin spectra of its summands, and that this result does not hold, in general, for the generalized B-Weyl spectrum.

Proposition 3.1. *Let $A \in L(X)$ and $B \in L(Y)$. Then*

$$\sigma_{gD}(M_0) = \sigma_{gD}(A) \cup \sigma_{gD}(B).$$

Proof. Let $\lambda \notin \sigma_{gD}(A) \cup \sigma_{gD}(B)$, then $\lambda \notin \mathrm{acc}\sigma(A) \cup \mathrm{acc}\sigma(B)$. As $\mathrm{acc}\sigma(A) \cup \mathrm{acc}\sigma(B) = \mathrm{acc}[\sigma(A) \cup \sigma(B)]$, then $\lambda \notin \mathrm{acc}\sigma(M_0)$. So $\lambda \notin \sigma_{gD}(M_0)$ and hence $\sigma_{gD}(M_0) \subset \sigma_{gD}(A) \cup \sigma_{gD}(B)$.

Conversely, let $\lambda \notin \sigma_{gD}(M_0)$, then $\lambda \notin \mathrm{acc}\sigma(M_0) = \mathrm{acc}\sigma(A \oplus B)$. Therefore $\lambda \notin \mathrm{acc}\sigma(A) \cup \mathrm{acc}\sigma(B)$. Thus $\lambda \notin \sigma_{gD}(A) \cup \sigma_{gD}(B)$. Hence $\sigma_{gD}(A) \cup \sigma_{gD}(B) \subset \sigma_{gD}(M_0)$. This finishes the proof. □

Example 3.2. Let $R \in L(X)$ and $T \in L(X)$. Let A be the operator defined on $X \oplus X$ by

$$A = \begin{pmatrix} 0 & T \\ R & 0 \end{pmatrix},$$

then $A^2 = \begin{pmatrix} TR & 0 \\ 0 & RT \end{pmatrix}$. Thus it follows from the above proposition that $\sigma_{gD}(A^2) = \sigma_{gD}(TR) \cup \sigma_{gD}(RT)$, which equals to $\sigma_{gD}(TR)$ by Corollary 2.11. Therefore $\sigma_{gD}(A) = \sqrt{\sigma_{gD}(TR)}$.

Remark 3.3. In general, the equality proved in Proposition 3.1 for the generalized spectrum does not hold for the pseudo B-Weyl spectrum. For this, let S be the unilateral unweighted shift on $l^2(\mathbb{N})$ and set $A = S$ and $B = S^*$. Since A and B^* have SVEP then $\sigma(A) = \sigma_{gD}(A) = \sigma_{pBW}(A) = \sigma(B) = \sigma_{gD}(B) = \sigma_{pBW}(B) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$, while $0 \notin \sigma_{pBW}(M_0)$.

Generally, the study of generalized Drazin invertibility for upper triangular operator matrices was firstly investigated by D. S. Djordjević and P. S. Stanimirović [10]. They proved in particular that

$$\sigma_{gD}(M_C) \subset \sigma_{gD}(A) \cup \sigma_{gD}(B) \text{ for every } C \in L(Y, X). \tag{3.1}$$

But this inclusion may be strict as we can see in the following example.

Example 3.4. Let $A = S$ be the unilateral shift on $l^2(\mathbb{N})$ and let $B = S^*$ and $C = I - SS^*$. Then M_C is unitary and hence we get

$$\sigma_{gD}(M_C) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \text{ and } \sigma_{gD}(A) \cup \sigma_{gD}(B) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\}.$$

The defect set $(\sigma_{gD}(A) \cup \sigma_{gD}(B)) \setminus \sigma_{gD}(M_C)$ has been studied very recently in [36], more precisely, it was proved that this defect is the union of certain holes in $\sigma_{gD}(M_C)$ which happen to be subsets of $\sigma_{gD}(A) \cap \sigma_{gD}(B)$. We will explicit in what follows the defect set $[\sigma_{gD}(A) \cup \sigma_{gD}(B)] \setminus \sigma_{gD}(M_C)$ by means of localized SVEP. This result will lead us to a necessary condition that ensures the equality desired (see Corollary 3.6 bellow).

Theorem 3.5. *For $A \in L(X)$, $B \in L(Y)$ and $C \in L(Y, X)$ we have*

$$\sigma_{gD}(M_C) \cup [\mathcal{S}(A^*) \cap \mathcal{S}(B)] = \sigma_{gD}(A) \cup \sigma_{gD}(B).$$

Proof. Let $\lambda \in (\sigma_{gD}(A) \cup \sigma_{gD}(B)) \setminus \sigma_{gD}(M_C)$. We can assume without loss of generality that $\lambda = 0$. Then M_C is generalized Drazin invertible and hence $0 \notin \text{acc}\sigma(M_C)$. Then there exists $\varepsilon > 0$ such that $M_C - \mu I$ is invertible for every $0 < |\mu| < \varepsilon$. Thus for every $0 < |\mu| < \varepsilon$, $A - \mu I$ is left invertible and $B - \mu I$ is right invertible. So $0 \notin \text{acc}\sigma_{ap}(A) \cup \text{acc}\sigma_s(B)$. For the sake of contradiction assume that $0 \notin \mathcal{S}(A^*) \cap \mathcal{S}(B)$.

Case 1. $0 \notin \mathcal{S}(A^*)$: If $0 \in \sigma(A^*)$ then since $\sigma(A^*) = \mathcal{S}(A^*) \cup \sigma_s(A^*)$ we have $0 \in \sigma_s(A^*)$. As $0 \notin \text{acc}\sigma_{ap}(A) = \text{acc}\sigma_s(A^*)$, then 0 is an isolated point of $\sigma_s(A^*)$. Thus 0 is an isolated point of $\sigma(A^*) = \sigma(A)$. Hence A is generalized Drazin invertible. Summing up: M_C and A are generalized Drazin invertible, which implies from [36, Lemma 2.5] that B is also generalized Drazin invertible. But this is impossible. Now if $0 \notin \sigma(A^*)$ then $0 \notin \sigma_{gD}(A)$. Hence B will be generalized Drazin invertible, and this is a contradiction.

Case 2. $0 \notin \mathcal{S}(B)$: If $0 \notin \sigma(B)$ then $0 \notin \sigma_{gD}(B)$. So B is generalized Drazin invertible, and since M_C is generalized Drazin invertible it follows from [36, Lemma 2.5] that A is generalized Drazin invertible. But this is a contradiction. If $0 \in \sigma(B)$

then $0 \in \sigma_s(B)$. As $0 \notin \text{acc}\sigma_s(B)$ then $0 \in \text{iso}\sigma_s(B)$, therefore $0 \in \text{iso}\sigma(B)$. So B is generalized Drazin invertible and since M_C is generalized Drazin invertible, it then follows that A is generalized Drazin invertible. But this is a contradiction.
In the two cases we have $\sigma_{gD}(A) \cup \sigma_{gD}(B) \subset \sigma_{gD}(M_C) \cup [\mathcal{S}(A^*) \cap \mathcal{S}(B)]$. Since the opposite inclusion is always true then $\sigma_{gD}(A) \cup \sigma_{gD}(B) = \sigma_{gD}(M_C) \cup [\mathcal{S}(A^*) \cap \mathcal{S}(B)]$. Hence the theorem is proved. □

Now, in the next corollary, we give a sufficient condition which ensures that $\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$ for every $C \in L(Y, X)$. We notice that the condition $\mathcal{S}(A^*) \cap \mathcal{S}(B) = \emptyset$ is not satisfied for operators A and B defined in Example 3.4.

Corollary 3.6. *Let $A \in L(X)$ and Let $B \in L(Y)$. If $\mathcal{S}(A^*) \cap \mathcal{S}(B) = \emptyset$ then for every $C \in L(Y, X)$, $\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$. In particular, if A^* or B has SVEP, then $\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$.*

Example 3.7. Let S be the unilateral shift operator on $l^2(\mathbb{N})$ and we define operators $A = (S \oplus S^*) + I$ and $B = (S \oplus S^*) - I$ on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$. Then

$$\sigma(A) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda - 1| \leq 1\}\ ,\ \sigma(B) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda + 1| \leq 1\}.$$

It follows that

$$\mathcal{S}(A) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda - 1| < 1\}\ ,\ \mathcal{S}(B) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda + 1| < 1\}.$$

Hence $\mathcal{S}(A^*) \cap \mathcal{S}(B) = \emptyset$. Therefore $\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$. Note here that A^* and B do not have SVEP.

The equality $\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$ holds in particular, if we take $A = S^*$ or $B = S$, since in this case A^* or B has SVEP. It also holds when A and B belong to the class of all normal or hyponormal operators in Hilbert spaces, or the class of all compact operators in Banach spaces.

Remark 3.8. Generally, we do not have $\sigma_{pBW}(M_C) = \sigma_{pBW}(A) \cup \sigma_{pBW}(B)$ even if A^* or B has SVEP. For instance, let S be the unilateral unweighted shift on $l^2(\mathbb{N})$. Let $A = S^*$, $B = S$ and $C = I - SS^*$. Since A^* and B have SVEP, it follows from Corollary 2.8 that $\sigma_{pBW}(A) = \sigma_{gD}(A)$ and $\sigma_{pBW}(B) = \sigma_{gD}(B)$. Hence $\sigma_{pBW}(A) \cup \sigma_{pBW}(B) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\}$. Since M_C is unitary then $\sigma_{pBW}(M_C) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Thus $\sigma_{pBW}(M_C) \neq \sigma_{pBW}(A) \cup \sigma_{pBW}(B)$. Here A and B^* do not have SVEP. However, we have the following result.

Proposition 3.9. *Let $A \in L(X)$ and $B \in L(Y)$. If A and B (or A^* and B^*) have SVEP, then for every $C \in L(Y, X)$,*

$$\sigma_{pBW}(M_C) = \sigma_{pBW}(A) \cup \sigma_{pBW}(B).$$

Proof. If A and B have SVEP then M_C has also SVEP, see [16, Proposition 3.1]. Hence

$$\begin{aligned} \sigma_{pBW}(M_C) &= \sigma_{gD}(M_C) \ \ \text{(by Corollary 2.8)} \\ &= \sigma_{gD}(A) \cup \sigma_{gD}(B) \ \ \text{(by Corollary 3.6)} \\ &= \sigma_{pBW}(A) \cup \sigma_{pBW}(B) \ \ \text{(by Corollary 2.8).} \end{aligned}$$

The case of A^* and B^* have SVEP goes similarly. □

In our next result, we are going to provide a new condition under which the equality $\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$ holds.

Proposition 3.10. *Let $A \in L(X)$, $B \in L(Y)$ and $C \in L(Y,X)$. If $\sigma_{pBW}(M_C) = \sigma_{pBW}(A) \cup \sigma_{pBW}(B)$, then $\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$.*

Proof. Let $\lambda \notin \sigma_{gD}(M_C)$ arbitrary, then $M_C - \lambda I$ is generalized Drazin invertible. Hence $\lambda \notin \sigma_{pBW}(M_C) = \sigma_{pBW}(A) \cup \sigma_{pBW}(B)$. So $A - \lambda I$ and $B - \lambda I$ are pseudo B-Weyl operators. If $\lambda \in \sigma_{gD}(A)$ then form Theorem 2.7 we have $\lambda \in \mathcal{S}(A) \cap \mathcal{S}(A^*)$. Hence $\lambda \in \mathcal{S}(A) \subset \mathcal{S}(M_C) \subset \sigma_{gD}(M_C)$. But this is a contradiction. Therefore $\lambda \notin \sigma_{gD}(A)$. From [36, Lemma 2.5] we conclude that $\lambda \notin \sigma_{gD}(B)$. Thus $\lambda \notin \sigma_{gD}(A) \cup \sigma_{gD}(B)$. Hence $\sigma_{gD}(M_C) \supseteq \sigma_{gD}(A) \cup \sigma_{gD}(B)$. Since $\sigma_{gD}(M_C) \subset \sigma_{gD}(A) \cup \sigma_{gD}(B)$ holds with no restriction, then $\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$. □

One might expect that the converse of Proposition 3.10 is true, but this is not true in general as shown in the following example.

Example 3.11. Let S be the unweighted unilateral shift on $l^2(\mathbb{N})$. On $l^2(\mathbb{N}) \otimes l^2(\mathbb{N})$ set $A = S \otimes I$, $B = S^* \otimes I$ and

$$C = \begin{pmatrix} 0 & & & \\ & I - SS^* & & \\ & & I - SS^* & \\ & & & \ddots \end{pmatrix}.$$

Then $\sigma(M_C) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\} = \sigma(A) = \sigma(B)$. Hence $\sigma_{gD}(M_C) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\} = \sigma_{gD}(A) \cup \sigma_{gD}(B)$. Since A and B^* have SVEP, then it follows from Corollary 2.8 that $\sigma_{pBW}(A) \cup \sigma_{pBW}(B) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$, while $\sigma_{pBW}(M_C) \subset \sigma_W(M_C) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

The following result gives necessary and sufficient condition under which the generalized Drazin spectrum of the operator M_C is the union of generalized Drazin spectra of its diagonal entries.

Proposition 3.12. *Let $A \in L(X)$, $B \in L(Y)$ and $C \in L(Y,X)$. Then the following assertions are equivalent.*
i) $\sigma(M_C) = \sigma(A) \cup \sigma(B)$;
ii) $\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B)$;
iii) $\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$.

Proof. For (i) $\Longleftrightarrow$ (ii) see [34, Proposition 3.7].

i) $\Leftrightarrow$ iii) was proved in [36] but we give here another proof by using the local spectral property SVEP. Assume that $\sigma(M_C) = \sigma(A) \cup \sigma(B)$. Then it follows from [13, Theorem 2.5] that $\mathcal{S}(A^*) \cap \mathcal{S}(B) \subset \sigma(M_C)$. Since $\mathcal{S}(A^*) \cap \mathcal{S}(B)$ is an open subset then $\mathcal{S}(A^*) \cap \mathcal{S}(B) \subset \text{acc}\,\sigma(M_C)$. Thus $\mathcal{S}(A^*) \cap \mathcal{S}(B) \subset \sigma_{gD}(M_C)$. By Theorem 3.5 we conclude that $\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$.

Conversely, suppose that $\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$. Then it follows from Theorem 3.5 that $\mathcal{S}(A^*) \cap \mathcal{S}(B) \subset \sigma_{gD}(M_C)$. So $\mathcal{S}(A^*) \cap \mathcal{S}(B) \subset \sigma(M_C)$. This implies again by [13, Theorem 2.5] that $\sigma(M_C) = \sigma(A) \cup \sigma(B)$. □

Corollary 3.13. *Let $A \in L(X)$, $B \in L(Y)$. If $\sigma(A) = \sigma_l(A)$ or $\sigma(B) = \sigma_r(B)$ then for every $C \in L(Y,X)$ we have*

$$\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B).$$

Proof. If $\sigma(A)=\sigma_l(A)$ or $\sigma(B)=\sigma_r(B)$ then $\sigma(M_C)=\sigma(A)\cup\sigma(B)$, see [3]. But this is equivalent from Proposition 3.12 to say that $\sigma_{gD}(M_C)=\sigma_{gD}(A)\cup\sigma_{gD}(B)$. □

For U and $V\in L(X)$, let L_U (resp., R_V) be the left (resp., right) multiplication operator given by $L_U(W)=UW$ (resp., $R_V(W)=WV$) and let $\delta_{U,V}=L_U-R_V$ be the usual generalized derivation associated with U and V. Let $\mathcal{N}^\infty(U)=\bigcup_{n\geq 1}\mathcal{N}(U^n)$ denote the generalized kernel of U.

Theorem 3.14. *Let $A\in\mathcal{L}(X)$ and $B\in\mathcal{L}(Y)$. If C is in the closure of the set*

$$\mathcal{R}(\delta_{A,B})+\mathcal{N}(\delta_{A,B})+\bigcup_{\lambda\in\mathbb{C}}\mathcal{N}^\infty(L_{A-\lambda I})+\bigcup_{\lambda\in\mathbb{C}}\mathcal{N}^\infty(R_{B-\lambda I}),$$

then

$$\sigma_{gD}(M_C)=\sigma_{gD}(A)\cup\sigma_{gD}(B).$$

Proof. If C is in the closure of the set $\mathcal{R}(\delta_{A,B})+\mathcal{N}(\delta_{A,B})+\bigcup_{\lambda\in\mathbb{C}}\mathcal{N}^\infty(L_{A-\lambda I})+\bigcup_{\lambda\in\mathbb{C}}\mathcal{N}^\infty(R_{B-\lambda I})$, then $\sigma_D(M_C)=\sigma_D(A)\cup\sigma_D(B)$, see [34, Theorem 3.4]. The result follows at once from Proposition 3.12. □

In general there is no definite relation between the condition considered in Corollary 3.6 and the condition considered in the above theorem. Indeed, let $A=S$ and $B=S^*$. Set $C=A-B$, then $C\in\mathcal{R}(\delta_{A,B})$. Hence $\sigma_{gD}(M_C)=\sigma_{gD}(A)\cup\sigma_{gD}(B)$ while $\mathcal{S}(A^*)\cap\mathcal{S}(B)=\{\lambda\in\mathbb{C}\,:\,|\lambda|<1\}$.

Now, in the following definition, we introduce the concept of right and left generalized Drazin invertibility for bounded linear operators.

Definition 3.15. Let $T\in L(X)$. We will say that
i) T is *left generalized Drazin invertible* if $0\notin\mathrm{acc}\sigma_l(T)$.
ii) T is *right generalized Drazin invertible* if $0\notin\mathrm{acc}\sigma_r(T)$.
The *right generalized Drazin spectrum* is defined by

$$\sigma_{rgD}(T)=\{\lambda\in\mathbb{C}:T-\lambda I\text{ is not right generalized Drazin invertible}\}$$

and the *left generalized Drazin spectrum* is defined by

$$\sigma_{lgD}(T)=\{\lambda\in\mathbb{C}:T-\lambda I\text{ is not left generalized Drazin invertible}\}.$$

Theorem 3.16. *Let $A\in L(X)$, $B\in L(Y)$ and $C\in L(Y,X)$. If M_C is generalized Drazin invertible, then the following statements hold.*
i) A is left generalized Drazin invertible.
ii) B is right generalized Drazin invertible.
iii) There exists a constant $\gamma>0$ such that $d(A-\lambda I)=n(B-\lambda I)$ for every $0<|\lambda|<\gamma$.

Proof. Assume that M_C is generalized Drazin invertible. Then there exists $\gamma>0$ such that $M_C-\lambda I$ is invertible for every $0<|\lambda|<\gamma$. So and by virtue of [14, Theorem 2] we have $A-\lambda I$ is left invertible and $B-\lambda I$ is right invertible for every $0<|\lambda|<\gamma$. Thus $0\notin\mathrm{acc}\sigma_l(A)\cup\ \mathrm{acc}\sigma_r(B)$. This proves that A is left generalized Drazin invertible and B is right generalized Drazin invertible. On the other hand, since $M_C-\lambda I$ is invertible for $0<|\lambda|<\gamma$, then again by [14, Theorem 2] we obtain that $d(A-\lambda I)=n(B-\lambda I)$ for every $0<|\lambda|<\gamma$. □

From Theorem 3.16, we derive the following corollary.

Corollary 3.17. *Let $A \in L(X)$, $B \in L(Y)$. Then for every $C \in L(Y, X)$ we have*

$$[\sigma_{gD}(A) \cup \sigma_{gD}(B)] \setminus [\sigma_{rgD}(A) \cap \sigma_{lgD}(B)] \subset \sigma_{gD}(M_C).$$

Proof. Let $\lambda \in [\sigma_{gD}(A) \cup \sigma_{gD}(B)] \setminus \sigma_{gD}(M_C)$, then $A - \lambda I$ is left generalized Drazin invertible and $B - \lambda I$ is right generalized Drazin invertible, by Theorem 3.16. If $\lambda \notin \sigma_{rgD}(A)$, then $A - \lambda I$ is generalized Drazin invertible. Since $M_C - \lambda I$ is generalized Drazin invertible, then $B - \lambda I$ is also generalized Drazin invertible. This is a contradiction. Analogously, we have $\lambda \in \sigma_{lgD}(A)$. Thus $\lambda \in \sigma_{rgD}(A) \cap \sigma_{lgD}(B)$. $\square$

The following theorem gives a slight generalization of the main result of [36].

Theorem 3.18. *Let $A \in L(X)$, $B \in L(Y)$. Then for every $C \in L(Y, X)$ we have*

$$\sigma_{gD}(A) \cup \sigma_{gD}(B) = \sigma_{gD}(M_C) \cup \mathcal{W},$$

where $\mathcal{W}$ is the union of certain holes in $\sigma_{gD}(M_C)$ which happen to be subsets of $\sigma_{rgD}(A) \cap \sigma_{lgD}(B)$.

Proof. From [36] we have

$$\eta(\sigma_{gD}(A) \cup \sigma_{gD}(B)) = \eta(\sigma_{gD}(M_C),$$

where $\eta(.)$ is the polynomially convex hull. By Corollary 3.17 the filling some holes in $\sigma_{gD}(M_C)$ should occurs in $\sigma_{rgD}(A) \cap \sigma_{lgD}(B)$. $\square$

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Centre régional des métiers de l'éducation et de la formation,, B.P 458, Oujda, Morocco et, Equipe de la Théorie des Opérateurs, Université Mohammed I, Faculté des Sciences, Dépt. de Mathématiques et Informatique, Morocco.
E-mail address: `h.zariouh@yahoo.fr`

Department of Mathematics and Computer Science, Multidisciplinary Faculty of Nador, Mohammed First University, PO Box 300, Selouane 62702 Nador, Morocco.
E-mail address: `h.zguitti@ump.ma`